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Approximately Straight Kink Theories

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A precise meaning is given to the idea of a kink theory approximating a vector- or vector-bundle-valued theory. It is shown that vector theories taking values in a vector bundle with group $SO(n-s,s;\mathbb{R})$, acting naturally, do not approximate any kink theory. It is further shown that, where a kink theory is approximated by a vector bundle theory, the field equations in the vector theory can give rise to field equations in the kink theory. The theory of Skyrme and the sine-Gordon theory are of this form. An example is given of a nonlinear modification of electromagnetism having solitonlike solutions.

1. INTRODUCTION

A kink theory is, roughly speaking, a field theory where the field takes values in a manifold that is not a vector space (or vector bundle). Such a theory was investigated from 1958 by Skyrme (1958; see also 1971), while the generalized idea of kink theory was introduced in 1959 by Finkelstein and Misner (1959). The concept is of interest, first, because some accepted field theories (e.g., general relativity) can be regarded as kink theories; and, second, because the particular subclass of kink theories to be considered here give a very simple way of incorporating nonlinearity into vector-valued ("straight") field theories and offers the hope of reaching rigorous quantized nonlinear theories.

While the motivation stems from the quantized versions of kink theory, it has proved necessary to treat the classical and quantum cases separately. Here the classical case will be described in a general context which brings out the way kink theories can arise from "curving up" a straight theory. Quantization will be discussed in a subsequent paper where heavy restrictions will be placed on the theory, for technical reasons, thereby obscuring the very natural geometrical setting given here.

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2. DEFINITIONS

We shall be concerned with a kink theory and a straight theory that are related to a common principal bundle $\xi = (P, \pi, M, G)$; where P is the space of the bundle, π is a projection from P onto space–time M and G is a Lie group acting on P on the right ($p \to pg$ for $g \in G$) whose action is simply transitive on each fiber $\pi^{-1}(x)$. A typical example is furnished by taking P to be the bundle L of all pseudo-orthonormal frames over space–time (with respect to the Lorentz metric of general relativity), with $G = \mathfrak{L}$, the Lorentz group.

We recall the definition of an associated bundle. If $T: G \to$ Homeo (X) is an effective action of G on a topological space X, then we define an equivalence relation on $P \times X$ by

$$(p, x) \sim (pg, T(g^{-1})(x)) \qquad (g \in G)$$

The fiber bundle $\xi^T = (E^T, \pi)$, where E^T is the quotient space

$$E^T := (P \times X)/\sim$$

with projection

$$\pi^T := [(p, x)] \to \pi(p) \in M$$

is called the *bundle associated to* ξ *by* T with fiber X.

A straight field theory associated with ξ is a theory about certain sections $\theta: M \to E^T$ (called fields) where T is a linear representation of G in a finite-dimensional (real or complex) vector space $V \equiv X$; so that ξ^T is in this case a vector bundle. For example, taking $\xi \equiv \lambda = (L, \pi, M, \mathfrak{L})$, the frame bundle, $V = \mathbb{R}^4$ and T the standard action of $\mathfrak{L}$ on $\mathbb{R}^4$ gives for E^T the tangent bundle $T(M)$, whose cross sections ϕ are *vector fields*.

A *kink theory*, in the most general sense, is a theory in which the fields are sections of a fiber bundle E over M whose fibers are not homotopically trivial. In Skyrme's theory (1978; 1971) $E = M \times S^3$ (a product bundle) and the fields are maps $\phi: M \to S^3$. In general relativity the basic field might be regarded as the metric, which is a section of the bundle of all second-rank covariant symmetric tensors over M with Lorentz signature (+ - - -), a subbundle of the bundle $T_S^{0,2} = T^*(M) \circledS T^*(M)$ of all covariant symmetric tensors. Although $T_S^{0,2}$ is a vector bundle, the subset of tensors with Lorentz signature is not: its fiber is homotopic to projective 3-space (Shastri et al., 1980).

A kink theory is of interest in the following two cases: (i) M, space–time, is homotopically nontrivial; or (ii) we require, as a boundary condition on the fields, that they can be extended to a fiber bundle $\bar{E}$ over $\bar{M}$, where $\bar{M}$ is a space containing M as a dense open subset and is homotopically nontrivial.

An example of (ii) is the case $M=\mathbb{R}^1\times\mathbb{R}^3$ (time $\times$ space), $\bar{M}=\mathbb{R}^1\times S^3$, corresponding to setting boundary conditions as the fields tend to an ideal point at spatial infinity adjoined to $\mathbb{R}^3$.

In these two cases there is the possibility that some sections (fields) cannot be deformed one into another; when this phenomenon occurs the theory is said to admit kinks (Finkelstein, 1966).

Suppose we are given a straight theory whose fields are sections of a bundle with space E^τ associated to ξ by a linear representation τ in a vector space V. A kink theory with fields in a bundle E will be said to *correspond to* E^τ if the following conditions hold:

C(i) $E=E^T$, where T is a 1–1 homomorphism of G into the endomorphisms of a Lie group K.

C(ii) There exists a linear isomorphism $\theta: V\to\mathbf{k}$ (where $\mathbf{k}$ is the Lie algebra of K), satisfying

C(iii) $T(g)_*\circ\theta=\theta\circ\tau(g)$.

Equivalently, in this case we shall say that E^τ *approximates* E^T, in a sense which we shall now explain.

A bundle satisfying C(i) will be called a *Lie group bundle*. It is easily verified that each fiber is a Lie group with multiplication given by

$$[(p,k_1)][(p,k_2)]=[(p,k_1k_2)]$$

Moreover, the Lie algebra of the fiber $\pi^{T-1}(x)=:K_x$, which we can identify with $T_{e_x}(K_x)$ (where e_x is the identity element of K_x) is precisely the fiber $\pi^{\mathbf{t}-1}(x)=:\mathbf{k}_x$ of the bundle $E^{\mathbf{t}}$ associated to ξ by the representation

$$\mathbf{t}: g\to T(g)_*:\mathbf{k}\to\mathbf{k}$$

of G. But we can regard $T_{e_x}(K_x)$ as a vector space "approximation" to K_x, near e_x; and so the bundle $E^{\mathbf{t}}=\cup_x T_{e_x}(K_x)$ is, in this sense, an approximation to E^T. Now, condition (iv) implies that $E^{\mathbf{t}}\simeq E^\tau$ (a vector bundle isomorphism), and thus it is reasonable to call E^τ also an approximation to E^T, the approximation being close near the identity section $\cup_x\{e_x\}$.

We are more or less forced into this definition of a kink theory "corresponding to" a straight theory if we require that the kink theory have sufficient structure to enable one to define field equations (as we shall do below) and also that there be a distinguished section e of E (here the identity section $x\mapsto e_x$) with the property that

$$\cup_x T_{e(x)}(\pi^{-1}(x))\simeq E^\tau$$

(which seems to be the only precise meaning that can be given to the idea of E approximating E^{τ}).

3. EXISTENCE OF KINK THEORIES

We shall construct several examples in Section 4 below. One can, however, ask whether for any given straight theory there exists a corresponding kink theory; i.e., whether any straight theory can be kinked. A negative answer is provided by the following

> *Theorem.* Let G be $SO(n-s,s;\mathbb{R})$ $(1\leqslant s\leqslant n-1)$ and let τ be the standard action of G on $\mathbb{R}^n$. Then there exists no kink theory corresponding to ξ^{τ}.

Proof. Suppose there were such a theory, ξ^T. If K, the standard Lie-group fiber of ξ^T, is not simply connected, then we can extend T to the covering space $\tilde{K}$ (by extending the representation $\mathbf{t}$ on the algebra), denoting the resulting theory by $\xi^{\tilde{T}}$.

For simplicity let us identify $\mathbf{t}$ with τ (i.e., we identify $\mathbf{k}$ with $\mathbb{R}^n$ by a choice of basis in $\mathbf{k}$). The action of G on $\mathbf{k}$ allows us to fix a quadratic form g on $\mathbf{k}$ of signature $(n-s,s)$ invariant under G (up to a constant). g then defines a pseudo-Riemannian metric on $\tilde{K}$ by left translation in $\tilde{K}$, which makes $\tilde{K}$ a space of constant curvature (since G acts on $\tilde{K}$ as a full group of isotropies). Since $\tilde{K}$ is simply connected, it must be one of $\tilde{\mathbb{S}}^n_s, \mathbb{R}^n_s, \tilde{\mathbb{H}}^n_s$, where $\mathbb{R}^n_s$ is $\mathbb{R}^n$ with metric g^n_s of signature $(n-s,s)$ and $\tilde{\mathbb{S}}^n_s, \tilde{\mathbb{H}}^n_s$ are the universal covering spaces of a connected component of

$$\mathbb{S}^n_s := \left\{x\in\mathbb{R}^{n+1}_s : g^{n+1}_s(x,x)=1\right\}$$

$$\mathbb{H}^n_s := \left\{x\in\mathbb{R}^{n+1}_{s+1} : g^{n+1}_{s+1}(x,x)=-1\right\}$$

(up to a scale factor) (Wolf, 1974, p.67). Thus the isometry group of $\tilde{K}$ is $\tilde{SO}(n+1-s,s)$, $E(n-s,s)$, $\tilde{SO}(n-s,s+1)$, where $E(n-s,s)$ is the pseudo-Euclidean group and $\tilde{SO}$ denotes the covering corresponding to the coverings of $\mathbb{S}$ and $\mathbb{H}$.

But $\tilde{K}$ acts on itself by left translation as a *normal* subgroup of isometries; whereas $SO(p,q)$ is simple. Thus the only possibility is $\tilde{K}=\mathbb{R}^n_s$.

Hence K is an identification space of $\mathbb{R}^n_s$ admitting G as a (globally defined) group of isotropies. Thus $K=\mathbb{R}^n_s$, and ξ^T is not a kink theory.

4. EXAMPLES

4.1 The Trivial Case. Suppose that $G=\{1\}$, $P=M\times\{1\}, T:1\mapsto id_K$. Then K can be chosen arbitrarily to yield a kink theory whose fields are maps $M\to K$, with an approximating straight theory with fields $M\to\mathbf{k}$. This is the situation for the sine-Gordon field ($K=\mathbb{R}/2\pi\mathbb{Z}, \mathbf{k}=\mathbb{R}$), Skyrme's S^3-theory (1971) [$K=SU(2), \mathbf{k}=\mathbb{R}^3$] and the general situation usually considered in kink theory (Finkelstein and Misner, 1959; Williams and Zvengrowski, 1977).

Since this case is so widely applicable, it might seem that most of this paper is quite otiose. And this is true at the topological level: most of the straight theories that are seriously considered by physicists work in bundles whose group is reducible to the identity: they are trivial bundles. But when we introduce a connection in P, as we do below in order to define field equations, then G will in general be the holonomy group of the connection and as such plays an essential role. In the language of particle physics, it provides the gauge transformations of the theory, and our definition of correspondence between straight and kinked theories requires that this gauge freedom be preserved. If G and τ are such that there is no corresponding kink theory (as in Section 3 above), but ξ^τ is trivial, then a kink theory can be introduced but only by breaking the gauge symmetry in order to reduce G.

We note that Skyrme's theory can also be viewed as a gauge theory with $G=SO(3), K=SU(2), T$ defined by requiring $\mathbf{t}(G)$ to be the group of isotropies at the identity of $SU(2)$ and $P=M\times G$ (Skyrme, 1971).

4.2. The Adjoint Construction. Suppose $G=K$. Then we can take T to be the adjoint action

$$\mathrm{Ad}(g)(k)=gkg^{-1}$$

If we take $\xi\equiv\lambda$ to be the pseudo-orthonormal frame bundle on space–time, with $G=K=SO_+(1,3)=\mathfrak{L}$ (assuming full orientability), then the fields in ξ^{Ad} are tensor-valued fields satisfying

$$\phi_\nu^{\mu} g_{\mu\rho}\phi_\lambda^{\rho}=g_{\nu\lambda}$$

The corresponding straight theory [since $\mathbf{k}=\mathbf{so}(1,3)=$skew matrices] has bivector-valued fields

$$\psi_{[\mu\nu]}=\psi_{\mu\nu}$$

(e.g., electromagnetism). The approximation between the two is expressed by

$$\phi_\nu^\mu \approx \delta_\nu^\mu + g^{\mu\lambda}\psi_{\lambda\nu}$$

5. COVARIANT DERIVATIVES

In order to introduce field equations we shall define a covariant derivative in ξ^T which reduces to the usual definition when T is a linear representation.

Proposition. Given $\xi^T = (E^T, \pi^T)$ associated to a principal bundle ξ with a connection, and a vector field $X \in C^\infty(M, T(M))$, then there exists a covariant derivative

$$\nabla_X^T : C^\infty(\xi^T) \to C^\infty(\xi^{\mathfrak{t}})$$

from the smooth sections of ξ^T into those of $\xi^{\mathfrak{t}}$ satisfying the usual axioms for a derivation.

Proof. Write X for some $X(x) \in T_x(M)$. Choose $C:[0,1) \to M$ with $\dot{C}(0) = X$ and choose any $p \in \pi^{-1}(x)$. Let C_p^* be the horizontal lift of C through initial point p. Given $\phi \in C^\infty(\xi^T)$, define $\psi_p:[0,1) \to K$ by

$$\phi(C(t)) = \left[\left(C_p^*(t), \psi_p(t)\right)\right] \in E^T$$

$\dot{\psi}_p(0)$ is in $T_{\psi_p(0)}(K)$; we regard it as an element of $\mathbf{k}$ and represent it in $T_e(K)$ (e = identity of K) by forming

$$v_p = L\left(\psi_p(0)^{-1}\right)_* \dot{\psi}_p(0) \in T_e(K) = \mathbf{k} \tag{5.1}$$

[where $L(a)(x) \equiv ax$ is left translation]. Define

$$\nabla_X^T \phi = \left[(p, v_p)\right] \in E^{\mathfrak{t}}$$

We must check that this is independent of the choice of p by altering to $p' = pg, g \in G$. Then

$$\phi(C(t)) = \left[\left(C_p{}^*(t)g, T(g^{-1})\psi_p(t)\right)\right] = \left[\left(C^*{}_p{}'(t)g, \psi_p{}'(t)\right)\right]$$

So

$$v_{p'} = L\Big(\big\{T(g^{-1})(\psi_p(0))\big\}^{-1}\Big)_* T(g^{-1})_* \dot{\psi}_p(0) \tag{5.2}$$

Now

$$\begin{aligned} L\Big(\big\{T(g^{-1})(\psi_p(0))\big\}^{-1}\Big)T(g^{-1})(k) &= T(g^{-1})\big(\psi_p(0)^{-1}\big)\cdot T(g^{-1})(k) \\ &= T(g^{-1})\big(\psi_p(0)^{-1}k\big) \\ &= T(g^{-1})\Big(L\big(\psi_p(0)^{-1}\big)(k)\Big) \end{aligned}$$

for all $k \in K$. Thus (5.2) becomes

$$\begin{aligned} v_{p'} &= T(g^{-1})_* L\big(\psi_p(0)^{-1}\big)_* \dot{\psi}_p(0) \\ &= T(g^{-1})_* v_p \qquad \text{from (5.1)} \\ &= \mathbf{t}(g^{-1}) v_p \end{aligned}$$

Hence $[(p', v_{p'})] = [(pg, \mathbf{t}(g^{-1})v_p)]$

$$= \big[(p, v_p)\big] \qquad \text{as required}$$

■

Corollary. If ξ^T is a kink theory corresponding to ξ^τ, then there exists a covariant derivative

$$\nabla_X^\theta : C^\infty(\xi^T) \to C^\infty(\xi^\tau)$$

Proof. Given $\phi \in C^\infty(\xi^T)$, define

$$\nabla_X^\theta \phi = \big[(p, \theta^{-1} v_p)\big]$$

where

$$\nabla_X^T \phi = \big[(p, v_p)\big]$$

This is independent of p by virtue of C(iii), Section 2.

The construction of higher derivatives is now straightforward. That is, we define $\nabla^\theta \phi : X \to \nabla_X^\theta \phi$ so that $\nabla^\theta \phi \in T^*(M) \otimes E^\tau$. This bundle is a vector bundle associated to the bundle $(\lambda \oplus \xi) = (L \otimes P, \pi, M, \mathfrak{L} \times G)$, with standard fiber $\mathbb{R}^4 \times V$, and $\lambda \oplus \xi$ acquires a connection from the Levi–Civita connection on λ and the given connection on ξ. Thus $T^*(M)\oplus E^\tau$ has a connection in the usual sense of vector bundles, and we can differentiate $\nabla^\theta\phi$ covariantly in the usual sense, forming $\nabla\nabla^\theta\phi,\ldots,\nabla^k\nabla^\theta\phi$.

6. FIELD EQUATIONS

We suppose given a principal bundle ξ with a connection, a straight theory ξ^τ (τ a representation of G in V) and a $\tau(G)$-invariant bilinear form b^V on V. The space–time M is furnished with its Lorentz metric g. The form b^V then induces a bilinear form b on the fibers of E^τ. In this situation the field equations typically assume the form

$$F(g_x, b_x, \psi(x), \nabla\psi(x), \nabla^2\psi(x),\ldots)=0 \tag{6.1}$$

($x\in M$), where $\nabla\psi, \nabla^2\psi$, etc. are covariant derivatives of $\psi\in C^\infty(\xi^T)$ defined relative to the connection on ξ and the Levi–Civita connection on λ, the frame bundle (as described in Section 5).

Now suppose we are given a corresponding kink theory. If F does not depend on $\psi(x)$, then we can immediately write down a corresponding field equation for the kinked field $\phi(x)$, namely,

$$F(g_x, b_x, \nabla^\theta\phi(x), \nabla\nabla^\theta\phi(x),\ldots)=0 \tag{6.2}$$

with $x\in M, \phi\in C^\infty(\xi^T)$. The important point is that, although (6.1) may be linear, (6.2) will not be linear in any sense.

For example, if we take F to be the function appropriate to Maxwell's equations (where b is given in terms of g)

$$F(g,\nabla\psi)=\begin{pmatrix} d\psi \\ \delta\psi\end{pmatrix}$$

where

$$(\delta\psi)_\lambda = g^{\mu\nu}(\nabla\psi)_{\mu\nu\lambda}$$

$$(d\psi)_{\lambda\mu\nu}=\tfrac{1}{3}\left[(\nabla\psi)_{\mu\nu\lambda}+(\nabla\psi)_{\nu\lambda\mu}+(\nabla\psi)_{\lambda\mu\nu}\right]$$

then (6.2), when applied to the kinked electromagnetism of Section 4.2 gives, on writing out the covariant derivatives and simplifying,

$$\phi^\nu{}_{\rho;\nu}=0\ ;\ \phi_{\alpha[\sigma}\phi^\alpha{}_{\nu;\mu]}=0 \tag{6.3}$$

The second equation is essentially nonlinear, the nonlinearity arising from the term $L(\psi_p(0)^{-1})_*$ in (5.1). [The semicolon in (6.3) denotes the usual covariant derivative, not ∇^T.] As with sine-Gordon theory, the nonlinearity in (6.3) is not a mere perturbation of the straight theory, but an indication of the existence of kinked fields that cannot be deformed away. This gives rise to solitonlike phenomena: our "nonlinear electromagnetism" has globally regular solutions that cannot die away with time. It appears that the simplest of such solutions should look like magnetic dipoles at spatial infinity, though no solution has yet been found explicitly.

Note that Skyrme's theory (1971) has the form (6.2), since his $B_{\alpha\mu}$ are our $(\nabla^T\phi)_{\mu\alpha}$ (with μ the derivative index and α indexing Lie algebra components).

If the field equations (6.1) involve ψ as well as its derivatives, there is no longer an immediate generalization to kinked theories. But in many cases there is still an eminently reasonable generalization. For example, consider again the case where $E^{\mathfrak{t}}$ consists of bivector fields and E^T of Lorentz-tensor fields; but now suppose the field equations for the straight theory are the Klein–Gordon equations

$$\left(g^{\alpha\beta}\nabla_\alpha\nabla_\beta\psi\right)_{\gamma\delta}-m^2\psi_{\gamma\delta}=0$$

A natural generalization of this to the corresponding kink theory is then

$$\left(g^{\alpha\beta}\nabla_\alpha^T\nabla_\beta^T\phi\right)_{\gamma\delta}-m\phi_{[\gamma\delta]}=0$$

in which the term $\phi_{[\gamma\delta]}$ is a component of the invariantly defined object $[d(\mathrm{Tr}\phi)]^\#$, where $\lambda\to\lambda^\#$ is the map from $T^*(\mathfrak{L}_x)\to\mathbf{so}_x(1,3)$ defined by the natural inner product on the Lie algebra $\mathbf{so}_x(1,3)$ of the fiber $\mathfrak{L}_x=\pi^{-1}(x)$ of E^{Ad}. This example, which easily extends to a kink theory with fiber $SO(p,q)$ or $SO(n)$, generalizes sine-Gordon theory for which the fiber is $SO(2)$.

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